THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5000 Analysis I (Fall 2015)
Makeup Quiz 2

1. (a) By using the ϵ-definition, show that $\lim_{n\to\infty} \frac{4n}{2n-3} = 2$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x \in \mathbb{Q}; \\ 2x & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

   By using the $\delta - \epsilon$ definition, show that $f(x)$ is continuous at $x = 0$.

2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the set of cluster points of S.

   (Remark: Show why each point of the collection you give is the a cluster point of S and why the other points are not cluster points of S.)

3. Let $\{x_n\}$ be a sequence of bounded sequence.

   Define $s_n := \sup\{x_k : k \geq n\}$ and $t_n := \inf\{x_k : k \geq n\}$.

   (a) Show that $\{s_n\}$ and $\{t_n\}$ are convergent sequences.
   Furthermore, if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L \in \mathbb{R}$, then $\{x_n\}$ converges to L as well.

   (b) Give an example of $\{x_n\}$ such that $\{s_n\}$ and $\{t_n\}$ are convergent, but $\{x_n\}$ is divergent.

4. Let K be a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is a continuous function.

   Show that $f(K)$ is also a compact subset of $\mathbb{R}$.